Lyapunov Stability and Attractors of Some Systems of Nonlinear Oscillators

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Lyapunov functions valid in the greater part of phase space were found for a system of nonlinear oscillators of an extended Van der Pol type. They yield a good estimate of the location of attractors. For a particular single oscillator the appropriately modified Van der Pol equation delivers an ellipse as limit cycle.

The Van der Pol equation is basic in the area of nonlinear oscillations (see, for example, [1]). It is unstable near the origin but possesses a stable limit cycle. Changing the sign in front of the first derivative makes the origin an attractor whose basin is bounded by an unstable limit cycle. We construct here arbitrarily large systems of an extended Van der Pol type for which Lyapunov functions leading to a good estimate of the location of the attractors can be found. Let us consider

$$\ddot{Y} + \varepsilon[(Y, AY)M + (\dot{Y}, B\dot{Y})N - P]\,\dot{Y} + C\dot{Y} = 0\,, \quad (1)$$

where Y is a real vector of arbitrary length N. A, B, C, M, N and P are $N \times N$ real matrices, whose properties will be specified later. $(\ldots, \ldots)$ denotes the scalar product and $\varepsilon = 1$ is taken first.

Assuming C to be symmetric and positive definite and taking the scalar product of (1) with $\dot{Y}$, one obtains

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}[(\dot{Y}, \dot{Y}) + (Y, CY)] \qquad (2) \\ &= -[(Y, AY)(\dot{Y}, M\dot{Y}) + (\dot{Y}, B\dot{Y})(\dot{Y}, N\dot{Y}) \\ &\quad - (\dot{Y}, P\dot{Y})]\,. \end{aligned}$$

Only the symmetric parts A_s, B_s, M_s, N_s, P_s of A, B, M, N, and P appear in (2). It is assumed that A_s, B_s, M_s, N_s and P_s are positive definite with highest eigenvalues α_1, β_1, μ_1, ν_1 and π_1 and lowest eigenvalues α_0, β_0, μ_0, ν_0 and π_0. Let the highest and lowest eigenvalues of C be denoted by γ_1 and γ_0.

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Two basic inequalities can now be derived from (2):

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}[(\dot{Y}, \dot{Y}) + (Y, CY)] \\ &\leqq -\beta_0\nu_0\Big[(\dot{Y}, \dot{Y}) + (Y, CY) + \frac{\alpha_0\mu_0}{\beta_0\nu_0}(Y, Y) \\ &\qquad - (Y, CY) - \frac{\pi_1}{\beta_0\nu_0}\Big](\dot{Y}, \dot{Y})\,, \qquad (3) \end{aligned}$$

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}[(\dot{Y}, \dot{Y}) + (Y, CY)] \\ &\geqq -\beta_1\nu_1\Big[(\dot{Y}, \dot{Y}) + (Y, CY) + \frac{\alpha_1\mu_1}{\beta_1\nu_1}(Y, Y) \\ &\qquad - (Y, CY) - \frac{\pi_0}{\beta_1\nu_1}\Big](\dot{Y}, \dot{Y})\,. \qquad (4) \end{aligned}$$

From inequality (4) we have instability around the origin and in the case

$$\gamma_0 \geqq \alpha_1\mu_1/\beta_1\nu_1 \qquad (5)$$

the instability persists if

$$(\dot{Y}, \dot{Y}) + (Y, CY) \leqq \pi_0/\beta_1\nu_1\,. \qquad (6)$$

From inequality (3) and

$$\gamma_1 \leqq \alpha_0\mu_0/\beta_0\nu_0 \qquad (7)$$

it can be seen that the system is stable if

$$(\dot{Y}, \dot{Y}) + (Y, CY) \geqq \pi_1/\beta_0\nu_0\,. \qquad (8)$$

If a plot is made in terms of $(\dot{Y}, \dot{Y})$ and (Y, CY), we know that any attractor will have to be in a strip between the two lines obtained from (6) and (8) (see plot).

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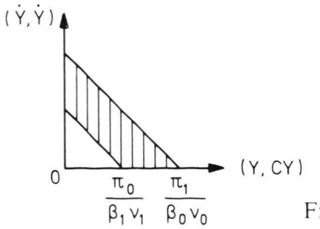

Fig. 1. Location of attractors.

An interesting case is when the strip goes to zero, i.e.

$$\pi_0/\beta_1 \nu_1 = \pi_1/\beta_0 \nu_0 . \tag{9}$$

According to the definitions, (9) can be verified only if

$$\pi_0 = \pi_1 = \pi \quad \text{and} \quad \beta_1 \nu_1 = \beta_0 \nu_0 = \beta \nu . \tag{10}$$

From (10) and the inequalities (7) and (5) we obtain

$$\alpha_1 \mu_1/\beta \nu \leqq \gamma_0 \leqq \gamma_1 \leqq \alpha_0 \mu_0/\beta \nu . \tag{11}$$

Inequalities (11) can only be verified as equalities, which means

$$\alpha_0 \mu_0 = \alpha_1 \mu_1 = \alpha \mu , \quad \gamma_0 = \gamma_1 = \gamma = \alpha \mu/\beta \nu . \tag{12}$$

Equations (10) and (12) are verified in particular if

$$\begin{aligned} A &= \alpha I , & B &= \beta I , \\ M &= \mu I , & N &= \nu I , \\ P &= \pi I , & C &= (\alpha \mu/\beta \nu) I , \end{aligned} \tag{13}$$

where I is the identity matrix. Equations (6) and (8) yield in that limit

$$(\dot{Y}, \dot{Y}) + \alpha \mu/\beta \nu (Y, Y) = \pi/\beta \nu . \tag{14}$$

Relation (14) annihilates the coefficient of $\dot{Y}$ in (1), which becomes

$$\ddot{Y} + (\alpha \mu/\beta \nu)\, Y = 0 . \tag{15}$$

Equations (14) and (15) are compatible and their common solutions yields a high-dimensional stable limit cycle. For $N = 1$ and conditions (13), (1) becomes

$$\ddot{y} + (\alpha \mu y^2 + \beta \nu \dot{y}^2 - \pi)\, \dot{y} + (\alpha \mu/\beta \nu)\, y = 0. \tag{16}$$

Equation (16) is an extension of the Van der Pol equation but has a much simpler limit cycle, i.e. an ellipse in the y, $\dot{y}$ plane. Note that the limit $\beta \nu \to 0$ to the Van der Pol equation is singular.

In the general case there is no reason to expect a limit cycle as attractor. More plausible is something of the sort considered in [2], sometimes called "strange" attractor. Numerical calculations in the strip in Fig. 1 may help to identify the attractor. Let us finally mention that it is straightforward to go through the arguments for $\varepsilon = -1$. In essence, the stable regions become unstable and vice versa, and the limit cycles become unstable. Note also that in essence all this theory still holds in case a positive monotonic nonlinearity $f((Y, HY)), f' > 0, H > 0$ is added in front of $\dot{Y}$ in (1) if at the same time the operator $f' H$ is added in front of Y.

[1] P. Hagedorn, Nichtlineare Schwingungen, Akademie Verlagsgesellschaft, Wiesbaden 1978.

[2] S. Smale, Differentiable Dynamical Systems, Bull. Amer. Math. Soc. **73**, 747 (1967).